THE DESIGN AND PROTOTYPE CONSTRUCTION OF A

COMPUTERIZED THREE-DIMENSIONAL

MODELING GAME

FOR THE DEVELOPMENT OF SPATIAL AND MATH SKILLS

by

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CHAPTER I:
INTRODUCTION
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Beginnings

When a lecturer at a teachers' workshop I was attending asked for ideas of how to use advanced computer technology in the classroom, I began to think about the current limits of middle school computer programming. I was teaching 3rd-8th grade computer classes in a parochial school, at the time. Also, my years of experience as a substitute in special education influenced me. After discovering students who were *very* bright intellectually, but unable to articulate their thoughts (such as students with learning disability, autism, or a certain type of low-verbal developmental disability), I saw the value of increasing their ability to communicate in a verbal way. Knowing that these students were often good at video games and manipulating geometric forms in their mind, I wondered how to draw them gradually toward a more commonplace vocabulary: traditional forms of notation (in math and answering story problems in words), with the use of computer programs.

Also, I realized that if this gap could be bridged, it would be of benefit to the development of regular students, as *all* of them are expected to crossover from concrete thinking to being able to construct a mental image of reality, enabling the creative analysis, combination, and restructuring of those symbolic forms.

Math would be a good choice of subject matter, both because of the universal need for basic math skills (individually and nationally) and because math already has a *very* traditional notation system: wire-frame outlines of geometric forms, with dashes to mark hidden lines, for example. It is the epitome of symbolic systems (Gauss' "queen of the sciences" (von Walters-Hausen, 1856)), connecting the worlds of reality and abstract thought, and allowing the student to easily manipulate symbolic forms that represent physical realities.

After seeing stunning animations of scientific visualization, resplendent in color, depth, and accuracy, I tried to think of ways to combine this with the usual programming options for children, which included Basic and Logo. As a pioneer, I searched for new areas for expansion, to be combined with processes that would enhance each individual student's intellectual and creative growth. From the 2-d computer programming language of Logo (where the children give a "turtle" simple directions to go forward, back, left, and right--and it appears on the *flat* screen of their computer monitor) I thought of expanding the frontier to include three dimensions and realistic-looking graphics.

In order to make it educationally useful, as well, I felt that it should fit in with existing trends in using computers to teach math, enhance the ability to see how math corresponds to reality (in space), and encourage creativity. I also foresaw a mixed hierarchical/floating structure of Nintendo®-like connections that would serve to engage the player as well as to present math in a logical sequence (when necessary). Optional modules could be added and deleted from the palette of games, depending upon the current interests and educational goals of the individual student.

Regarding management, the teacher's role would be assessment of the child's progress and interests to decide which games to make available. Automatically generated computer logs could record answers and (possibly) time spent, enabling the teacher to make informed decisions. Additionally, there would be capabilities for creative, technically-involved teachers to assemble their own games easily, following the invention of a child-controllable interface (based on the use of a 3 x 3 x 3 grid cube which could be layered over a more intricate architectural/computer graphics system).

Education and TV: Adversaries or Helpmates

Parents have long been concerned with getting their children to do homework instead of turning on the TV or a video game. Is this a monster in their living room? Statistics show that American youth stands out, but in a negative silhouette: they watch more television per day and do less math homework than many other countries. *Negative correlations have been shown to occur between TV-watching hours and high math scores.* Ten out of the 15 comprehensive populations of 13 year-olds represented in the 1991 International Assessment of Educational Progress (IAEP) survey showed this effect. The more TV was watched, according to three categories: 0-1, 2-4, and 5 or more hours per day, the lower the math scores were. Only two incomplete populations showed a positive correlation (perhaps due to *educational television* programs), and the highest scorers in those two countries only averaged about 50% correct. In the 9 year-old category, the top four (out of 10) comprehensive populations did not have a heavy television-watching percentage. The United States registered 26% of the younger students in the heaviest TV-watching category (Lapointe, Mead, & Askew, pp. 70, 94-95).

Besides the appearance of this high TV-watching/low math score correlation, an *overall downward trend in U.S. math achievement* has appeared, in recent decades. Also highlighted in the IAEP survey, 13 year-olds in the United States placed 14th among the 15 comprehensive populations included in the comparison; 9 year-olds, 9th out of 10th. International status on math scores has alarmed both math educators and parents. The top students are doing well, but the rest of the children lag behind *many* other countries.

The creation of math software that is fun to use might (hopefully) help to improve these statistics: Instead of being a threat, children's love of the TV (and video game) screen could be turned to the service of math education. To this end, I have envisioned an educational computer game set that would be mathematically accurate and graphically beautiful, making use of recently available 3-d dynamic computer graphics displays. Other authors have designed successful math software, but more is needed that will make use of the amazing technology available, children's interest in video games, spatial cognition, realistic representations, 3-d coordinates, and creativity. The envisioned software is not meant to supplant good teaching methods and the necessity for homework, but instead to be integrated with regular classroom instruction.

Games and Intelligent Tutoring Systems

Consideration of a design that is engaging to children is of utmost importance. Otherwise they won't spend time with it. Combining this with the beautifully intricate pathways of a gradually evolving intelligent tutoring system, should result in a "game" that is both interesting and geared to teaching on an individual level.

Even exercises don't have to be boring and repulsive; witness Mozart's keyboard exercises (the Piano Sonatas) that have become envigorating concert fare. Although fascinating details should not become distracting decorations, they *could* be harnassed to direct the child's attention in the right direction. Malone's theories about making games "fun" and some of the principles behind intelligent tutoring systems will be discussed in Chapter 2.

Functional Design and Patterns

The computer puzzles included in this study form part of a large, gradually gathering game set called "FunFunctions," which is designed to teach math in a way that is ***functional***:

(1) in the sense of being an *everyday*-type occurrence (for instance, riding in a car);

(2) in that *the game could actually be extended to be useful in the professional world* (for example, civil engineers construct real roads using mathematical formulas);

(3) because the *spatial skills emphasized* by these games would be *indirectly useful in many of the current* (and often newly developed) *professions* (such as CAD architectural design); and

(4) in that *mathematical functions* (such as graphing of functions) should be included in each of the games, since functions are at the core of math;

(5) it provides multiple representations. This last concept is especially important to this study: children are to be taught rules, not merely to collect facts. *Rule-based learning* should be

emphasized over pebble-collecting unrelated facts (by using multiple representation).

<u>**Multiple Representations**</u>

There are several reasons for using multiple representations of mathematical concepts, according to the ***Professional Standards for Teaching Mathematics*** (National Council of Teachers of Mathematics [NCTM], 1991):

(1) *math is made up of patterns, which the teacher demonstrates, so that the students can construct a model of understanding in their own minds* -- "learning occurs as students actively assimilate new information and experiences and construct their own meanings...each student's knowledge of mathematics is uniquely personal" (NCTM, 1991, p. 2);

(2) *emphasize ideas, not just correct answers* -- "In order to establish a *discourse*[1] that is focused on exploring mathematical ideas, not just on reporting correct answers, the means of mathematical communication and approaches to mathematical reasoning *must be broad and varied"* (NCTM, 1991, p. 52). In Standard 8, the authors mention the use of "patterns and functions to represent and solve *problems*" (NCMT, 1989, p. 98);

(3) *make connections* -- "The acquisition of mathematical concepts and procedures means little if the content is learned in an isolated way in which connections among the various mathematical topics are neglected" In Standard 4, the authors state that an ideal teacher "represents mathematics as a network of interconnected concepts and procedures; emphasizes connections *between mathematics and other disciplines* and connections to daily living" (NCTM, 1991, p. 89);

(4) *show the value of math in society and in other disciplines* -- "Students should see math as something that permeates society and, indeed, their own lives" (NCTM, 1991, p. 90).

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[1] NCTM defines "discourse" as "the ways of *representing*, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in those tasks" (NCTM, 1991, p. 20).

Instructional Ideas

How is this to be done, more specifically? The ***Standards*** mentions a broad range of possible representations: "various means for communication about mathematics should be accepted, including drawings, diagrams, invented symbols, and analogies...also help students learn to use calculators, computers, and other technological devices as tools for mathematical discourse" (NCTM, 1991, p. 52). It recommends that the teacher should *emphasize how the multiple representations are linked* "with the intent of expanding students' understanding of mathe-matical content and connections"(p. 89). Also, it states that *conventional notation should be introduced* "at points when doing so can further the work or the discourse at hand"(p. 52).

More specifically, the ***Curriculum and Evaluation Standards*** (NCTM, 1989, pp. 98-99), states that in the middle years (the initial focus of this study):

> ...the study of patterns and functions should focus on the analysis, representation, and generalizaion of functional relationships. These topics should first be explored as informal investigations. Students should be encouraged to observe and describe all sorts of patterns in the world around them: plowed fields, haystacks, architecture, paintings, leaves on trees, spirals on pineapples, and so on. As the students mature, instructional efforts can move toward building a firm grasp of the interplay among tables of data, graphs, and algebraic expressions as ways of describing functions and solving problems.

Three other fields have pertinent additions: Cognitive scientists/psychologists have often studied the use of representations. Besides using (1) *external models* (for example, a diagram on paper) to represent (2) *reality* (a sensory experience for the individual), they also mention (3) *intermediate representations* which are meant to bridge the gap between the known and unknown (such as when a student is first learning to use algebraic symbols, a balance scale represents equality). Also, successful experts may leave out steps in problem solving, because they have internal representations that are neither on paper nor shown in the concrete problem. This restructuring of information into larger units has become automaticized. The "chunking" together of information may be part of a third, invisible representation that is unexpected to a beginner (such as imagining vector forces in physics problems). Placing a problem into this standard, experts' form of representation is equivalent to translating it into a *canonical form*. Choosing a successful intermediate form involves finding a model that the student can relate to personally and yet is a good analogy for the original problem. Also, the size of the "chunks" must be suitable for the student's present level of advancement (for instance, being able to learn a many-digit number must be preceded by learning how to learn *small* number sequences).

Secondly, learning-style studies have shown that students favor certain modalities. If one, for example, has trouble listening to spoken explanations, information in another mode may be easier to recall. A lecture may be supplemented with hands-on activity, such as constructing a visual-tactile geometric figure out of straws. *Using more modes means the teacher has more opportunities to reach everyone,* especially if one is limited in their communications. For instance, an student with a learning disability may be able to replay an audio tape (to study), but have trouble with writing or studying notes, or vice versa. Enabling him/her access to different modalities may fill in gaps heretofore left as stumbling blocks to progress in math.

A third reason is given by cognitive-behaviorists, who say that multiple representations give one more of a chance to learn the definition of something. By giving a student many examples, the range of acceptable, specific items will better delineate the "imaginary" conceptual range. *The concrete is used to show the range of the abstract concept.* (A negative example is often used to emphasize the location of the boundaryline between what is and what is not to be included in a certain category of objects or behaviors. Besides illustrating the category of "chairs" with many varieties of chairs, the teacher might want to include at least one example of a nonchair.)

<u>Using Multiple Representations to Teach Math: An Example</u>

In a creative, supplementary teaching activity, a math lesson might be based on the introduction of *space figures* (three dimensional geometric forms) to 5th graders. In this case, tetrahedrons would be introduced.

Linked representations could be shown to the child, between:

(1) solid, realistic models of mountains (*tiny scale models* the child can hold and place on a table in various arrangements);

(2) flat, *2-d drawings* of several views of the mountains;

(3) a colorful *computer simulation* of walking in a simple path through the mountains; and

(4) wire-frame *diagrams that are mathematically correct.*

Procedure.

A. The child is allowed to arrange the tiny mountains on a table. As they are doing this, the teacher mentions the change in perspective as one moves around the table. (This part of the activity is only necessary for less advanced students.)

B. A computer screen shows a flat, 2-d drawing of three mountains. Each is in a different color or (if in black and white) a different pattern.They overlap each other, to show depth.

C. The child is asked to imagine walking a certain path through the mountains. Then, he is given a choice of potential viewpoints, as his final destination. Feedback is given by the computer.

D. Gradually, the child progresses to wire-frame diagrams that become more and more like correct mathematical tetrahedrons (in 3-d) than mountains.

E. If the *individual* is having trouble, or would like to explore (as a reward), the computer simulation of walking along the given path is shown, then the choices are given (of the destination's viewpoint). Also, helpful hints, usually short sentences, are given if the student makes certain errors.

F. Individual learning styles can also be compensated for by choosing one of three strands: TEXT, DIAGRAM, or ANIMATION, at the start. (This doesn't cover all modalities, but gives more than the usual *lack* of choice). Sometimes, a preferred modality may be used to lead the student into a second form of representation, once the initial display is understood. In this ricocheting way, students may gradually learn standard mathematical notation, and its relevance to the everyday world.

Realism and Cognitive Apprenticeship

By placing the example in a realistic situation (not a just text-book word problem), the student has more chance of being engulfed by that (intellectual) culture, and actually learning what a professional does. Brown, Collins, and Duguid (1989) state that "knowledge is situated, being in part a product of the activity, context, and culture in which it is developed" (p. 32). Further, they say that the way that school teaches the use of conceptual tools is often very different from the professionals use of them. One may be good at schoolwork, but not succeed in a career, and vice versa.

They suggest the use of "authentic" activities -- "most simply defined as the ordinary practices of the culture" (Brown, Collins, & Duguid, p. 34) to replace the "hermetically sealed" present-day school methods. Instead of word problems which have been in a recognizeable form for the last 500 years that is "foreign to authentic math practice," situated learning or "cognitive apprenticeships" should take place. Therefore, construction of a more "situated" environment would be of great benefit. The realism of recent advances in technology, game-like but mathematically and architecturally accurate constraints, and the use of an intelligent tutor[2] that is adhesive to the students' progress are three major breakthroughs that may be gained by using available computer technology.

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[2]*Intelligent tutoring system [*formerly called a tutorial*]:* Programs through which the computer assumes total responsibility for instruction. Tutorial programs are characterized by a dialogue between the student and the computer in which the direction and level of the dialogue are shaped by student input. Ideally, such programs would be high fidelity simulations of the best teaching behavior associated with a given topic and would control the variables associated with the what, why, when, and who of the instructional episode. (NCTM, 1981, p. 13)

Statement of Purpose: Curriculum Development

Construction of A Game Set to Increase Math/Spatial Comprehension

Spatial abilities have been divided into two main categories, that of spatial *visualization* and spatial *orientation*. These will be described in more detail later. Spatial visualization, especially, has been positively linked with math achievement. Doug Grouws, in the ***Handbook of Research on Mathematics Teaching and Learning*** (1992, p. 455), states that "mathematics achievement generally correlates with spatial visualization in the range of .30 to .60," mentioning several studies which support this: (Battista & Clements, 1990; Ben-Chaim, Lappan, & Houang, 1988; Fennema & Tartre, 1985; and Tartre, 1990). Although research has found a high correlation between spatial and math skills, exact details of this correlation have not been resolved yet. Also, some methods for the training of spatial skills have been developed, but are still a scarce commodity.

This study agrees with past research stating that: (1) spatial aptitude can be improved; (2) spatial skills can transfer to math skills; and (3) software can be developed to improve spatial aptitude. The accompanying paper describes construction of a prototype game set and gives reasons to support its usefulness (through theory, literature review, and an exploratory study).

Benefits

Some of the benefits of creating courseware[3] to supplement math classes are:

(1) interactivity--causing higher student motivation and a closer match to their individual skills;

(2) emphasis on the visual mode--helpful for those who have high spatial abilities;

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[3]*Courseware* is a set of computer programs with which a student has direct interaction during a learning sequence. The use of the term *courseware* is much more restrictive than *software*...since software also refers to management or utility programs with which the learner has no direct interaction. (NCTM, 1981, p. 4)

(3) a doorway between regular textbook geometry and the real world (especially helpful to this age group, which is in the middle of developing their abstract thinking abilities);

(4) insistence upon the use of multiple-modes, which will give variety, reinforce concepts with multiple examples, and may be used to lead the student toward standard math notation and reading.

Prototype

The purpose of this study is to design, test, and further shape this prototype courseware (and related instructional materials), which are specifically designed to improve spatial and related math skills. This paper describes construction of a prototype game set and gives reasons to support its usefulness.

Three of the games (Mountain/Perspectives, Mountain/Cubes, and Mountain/Angles) have been completed and have undergone limited testing and evaluation, while the fourth (Freeway) is in a preparatory stage, awaiting proof that its patchwork design is possible. Concrete details are provided as to its construction (which will hopefully take begin in 1994). Underlying it is the forthcoming invention of a custom-generated landscape (data) set, to be traversed with the use of a child-controlled interface. This interface might act as a shell, allowing other teachers to create their own spatially-related modules.

CHAPTER 2:
REVIEW OF THE LITERATURE
Intuition, Logic, Abstraction, and Intelligent Systems

Microworlds

Although the modeling of authentic environments is sought by the author, a truly realistic simulation would be overwhelming to a player's senses, because of the amount of detail presented. This would make it hard for the pupil to pinpoint important systematic properties and relationships that we are teaching. Therefore, certain themes are to be selected for inclusion in this small world. These themes would be emphasized verbally and visually, as well as by the existence of certain constraints in this artificial realm. Meanwhile, other, equally common characteristics would have to become invisible. This sort of learning environment has been called a "microworld."

There are three main categories of perceptual cues to be emphasized in this learning world. First of all, there is the emphasis on the basic themes, truths, or facts of the discipline being taught (in this case mathematics). Secondly, there is a presentation of the expert's conceptualization of this world. It may or may not be related to the visible world (as in representing vector forces as arrows). The third usage, here, is in the creation of intermediary forms that will engage the learner's interest and move him/her along to the next stage, ultimately to the expert's viewpoint. The stages are not necessarily sequential, although some of them are, depending upon the content area being taught. (These three categories are concerned with domain (or content area), the expert, and intermediate representation).

Interwoven throughout this microworld are two textures of 3-d realism and beauty. Apparently, adherence to graphic realism is more important to the beginning learner than to advanced students. According to Richard Burton: "...the displays in which information is given must resemble the real equipment...As the students learn more, this reliance decreases" (Burton, 1988, p. 122; Johnson, 1987). Increasingly abstract representations may be used, in keeping with the student's ability to organize his knowledge into more sophisticated structures.

Fidelity and Mental Models

Burton says that at least four kinds of realism, or *fidelity*, have been identified by researchers:

(1) *physical fidelity* - same sense of touch;

(2) *display fidelity* - same appearance;

(3) *mechanistic fidelity* - same behavior; and

(4) *conceptual fidelity* - is thought of as the same thing (Burton, p. 120-1).

The importance of each of these depends upon the stage that the learner is in. Based upon cognitive science's theories about learning, that a student's conception of a problem changes in stages as his comprehension develops ("learning is more like a series of miniparadigm shifts," Brown & Burton, 1987, p. 71), it would be best to give the student several graduated instructional models, rather than to teach the expert's model in the first step. Fischer, Brown, and Burton have proposed a framework that would be a progression of "increasingly complex microworlds that provide intermediate experiences such that within each microworld the student can see a challenging but attainable goal" (Burton, p. 123).

One of the problems involved would be to determine the student's present level. The kind of computer system that could do this would have to react to the student's answers and decide what screen to present next. Behind the scenes, it would have to keep track of the student's progress, find the difference between the student's and the expert's viewpoint, know when and how to help the student, and decide which problems to present next. This individualized and interactive system, based on cognitive science/psychology research, may be called an Intelligent Tutoring System (or ITS). Some examples of this may be seen in CMU's mathematics and programming tutors (Koedinger & Anderson, 1993).

Educational Philosophies and Definition of Terms

Certain educational philosophies and terms have arisen in conjunction with this technological development (Burton, 1988, pp. 110-111). Many of them are mentioned above:

(1) *constructivism* - learning actively; constructing knowledge out of the student's present concepts, by letting him/her "rediscover" new relationships, procedures, and facts; Brown and Burton wrote that: "Speaking metaphorically, unless the student's conceptual eyeglasses are more or less attuned to the world view or ontology of the simulation designer, then what appears as a meaningful event to the designer may be seen either as noise or not seen at all by the student" (Brown & Burton, 1987, p. 70).

(2) *rule-based learning* - learn with meaning; learn *procedures,* relationships, not just unlinked facts. "One of the longstanding debates in mathematics education," wrote Hiebert and Carpenter, "concerns the relative importance of conceptual knowledge versus procedural knowledge or of *understanding versus skill*...which kind of knowledge is most important is the wrong question to ask...A better question to ask is how conceptual and procedural knowledge are related" (Hiebert & Carpenter, 1992, p. 77);

(3) *bugs* - locate student misconceptions and correct them with the diagnostic help of the computer (see "Student Modeling");

(4) *connections* - (actively stressed in the NCTM ***Standards***) means that the teacher emphasizes textbook problems about the real world (see "cognitive apprenticeship," page 9); as well as parallels and interwoven areas, bridging separate disciplines;

(5) *metaskills* - the student must learn to be objective about his/her own style of learning; self-monitoring and -management;

(6) *manipulatives* - in order to encompass the student's symbolic learning within our "concrete" reality ("an intermediate 'pedagogical notation' bridges from the concrete material to the formal symbolic system"; learn actively; may also include computer usage (Kaput added that there must be a "balance between the use of physical materials and computer analogs of those materials" (Kaput, p. 528-529)).

Clements and Battista (1986, p. 29) wrote that, " The use of manipulatives ahould promote the development of *spatial* visualization...that underlies geometric thinking...[and] should be oriented toward problem solving."

(7) *multirepresentation* - model symbolic concepts in several different forms and modalities. Clements and Battista set a suggested guideline: that "Students should be involved with four representations of a new idea" (Clements & Battista, 1986, p. 29);

(8) *problem solving* - also related to rule-based learning: the student must learn methods of structuring and searching his knowledge, not just adding factual pebbles to his memory. Join the hunt for "the Ghost of General Transfer" mentioned by Singley (1989, p. 25): "A variety of researchers have recently called for the identification and codification of 'general' cognitive skills. Simon (1980) claimed that powerful general problem-solving methods do exist, and what's more, they can be taught." Others have criticized this view, saying, "neither mathematics education nor cognitive psychology has yet come up with a reasonable theory of problem-solving instruction, let alone any prescriptions for instruction which have broad application to mathematics classrooms"(Lester, 1982, p. 58).

(9) *Intelligent Tutoring System (ITS) or (ICAI).* "A computer program that (a) is capable of competent problem solving in a domain, (b) can infer a learner's approximation of competence, and (c) is able to reduce the difference between its competence and the student's through application of various tutoring strategies" (Polson, Richardson, & Soloway, 1988, p. 263).

Philosophical Roots

What has been the philosophical evolution of these ideas? Although there have been many philosophers, educators, scientists, mathematicians, and psychologists who polished these ideas over the centuries, there is room to mention only a few examples. A couple of philosophical themes have repeatedly surfaced in the long-running conversation (between major intellects) that has shaped our culture and our view of mathematics. For instance, René Descartes, a rationalist, argued that the concept of space arrives as an innate idea *before* experience, while Berkeley, an empiricist, believed that sensation must precede perception of reality (Hart & Moore, 1973, p. 252).

Blaise Pascal discussed the difference between *the mathematical mind* and *the intuitive mind* at the beginning of his ***Pensées*** (1670): The mathematical mind sees principles easily, but it is difficult to turn one's mind in that direction, while the intuitive mind sees easily the ordinary things that are in front of him. When the mathematicians are not intuitive, they want to have definitions *first*, before observing. He liked the geometers' logic best, because they combine the best of both worlds: they *don't* try to define certain *primitive* words, but they *do* carefully define everything else (in ***On Geometrical Demonstration***, 1657).

Henri Poincaré divided mathematicians into two categories, those with analytical minds (that is, seeking solid step-by-step progressions) and those with intuitive minds (using images, and leaping to conclusions). He defined intuition in a slightly different way, referring to the use of internal imagery, beyond Pascal's inclusion of direct observation. Combining both intuitive and analytical methods, Poincaré wrote that "A great advantage of geometry lies in the fact that in it the senses can come to the aid of thought, and help find the path to follow, and many minds prefer to put the problems of analysis into geometric form" (Poincaré, 1946, p. 380).

Ernst Cassirer theorized that there were three levels of spatial experience: (1) organic or active space, (2) perceptual space (in higher animals), and (3) symbolic or abstract space. He called the third level the borderline between human and animal worlds, saying that only humans have the ability to understand and represent abstract space, the space of "pure intuition," even without a concrete referrant (Hart & Moore, p. 252).

Beauty

Another interesting theme emerges with the separation of spatial perception into three parts: the fascinating elegance of mathematical ideas. The relevance of this theme should be considered in mathematics education. In ***The Foundations of Science***, Poincaré pointed out the importance of the "***intellectual beauty*** which hides beneath sensuous beauty...which makes intelligence sure and strong" (p. 368). He said that mathematics "has an esthetic aim...its adepts find therein delights analogous to those given by painting and music. They admire the delicate harmony of numbers and forms; they marvel when a new discovery opens to them an unexpected perspective...joy they thus feel...even though the senses take no part therein" (Poincaré, p. 280).

Math gives a different appearance to those who love it. What do these people see in math? Jerry King wrote in ***The Art of Mathematics*** (1992) that, "There is, first of all, the motivating force for mathematics which is *beauty*, and then the goal of mathematics which is *truth*. And finally, there is the importance of mathematics resulting from what the mathematical truths tell us about *reality*" (King, p. 8). The truth and reality descriptions may be self-evident, but what about the aspect of beauty?

There is a sense of beauty in both visible and non-visible worlds of math. In the visible, there is the calligraphy of math notation (as in the $\int$ - integral sign) and geometrical forms which represent the beauty of naturally-occurring shapes. In the second realm, there is the elegance of optimal formulas, patterns in nature, proofs, and surprising problem solutions. In between, there is the halfworld of scientific visualization, where visual models of physical data sets enable one to quickly recognize (otherwise hidden) patterns in the numbers. Scientific models can be attractive, fascinating, and revealing: Edward Tufte wrote, "To envision information--and what bright and splendid visions can result--is to work at the intersection of image, word, number, art...The principles of information design are universal--like mathematics--and are not tied to unique features of a particular language or culture" (Tufte, 1990, pp. 9-10).

On the next page is an example of scientific visualization which accurately pinpoints geographical distribution, chemical composition, and concentration of Los Angeles *smog*. Surprisingly, it is beautiful. Also, it shows many characteristics of the pollution, instantiously, that would not be evident after many days of poring over long lists of data in tables (although this is an accurate representation of *exactly* the same statistics). Characteristics of the image: the color is related to ozone concentration and the three-dimensional "view" appears to be from a plane. In an extended video version of this display, the clouds of pollution roll in, and eastward, in their daily cycle.

[put figure of supercomputer
modeling here]

Figure 1: Scientific visualization, with the use of a computer.
From ***Projects in Scientific Computing, 1989-90***. (1990, p. 8). Pittsburgh, PA: Pittsburgh Supercomputing Center.

King hoped that *all* students can learn to appreciate the mathematical elite's viewpoint, "provided the methods of presenting and writing mathematics are changed so as to bring the aesthetic component out into the open and to develop a sensitivity for it in students as we now attempt to develop in them a sensitivity for the poetry of Shakespeare, the music of Brahms..." (King, p. 134). How can one affect a student's development in this way? NCTM's ***Curriculum and Standards*** (1989, p. 98), suggests: "Exploring patterns helps students develop mathematical power and instills in them an appreciation for the *beauty* of mathematics." Once again, NCTM is suggesting searching for patterns, learning how to generalize them, and the multiple representation of functions. *Beautiful though accurate models* of these patterns will hopefully place a metaphor in the minds of the student--of the beauty of the patterns behind the data. Also, representing data over time or multiple examples, in *animation*, may help to emphasize the abstract idea behind the data. Certainly, recent advances in computer graphics have made both of these possibilities for mathematics instructors. Following this, we will discuss Piaget's view of the child's cognitive development. Next, are some suggestions about methods of instruction.

Mental Development

Jean Piaget described several stages that a child passes through in mental development. Beginning with reflexive actions (0-2 years -- sensorimotor) the child explores and experiments. In the next stage (about 2-8 years -- preoperational), the child learns to use images and languages, "and begins *construction* of its own universe" (Hart & Moore, p. 254). Then, the following period (7-12 years -- concrete operations), the child can think logically, and see things from different orientations. Gradually, (after about 13 -- formal operational thought) the images are able to take on a life of their own, separate from concrete reality, and the child is able to deal with abstractions, unconnected to the physical world. This is not by *exact* ages, but is rather an approximation. Hart and Moore wrote that this development is defined as "changes in levels of organization independent of time" (p. 255).

Also, many parts of the theories, such as whether the stages occur suddenly or gradually build up; are composed of grouped abilities or interwoven single threads; have become controversial. Catalysts and the effect of the child's environment have also been debated. However, the important theme here is the existence of *order* or sequence. And that it encompasses the double worlds of sensation-logic; or concrete-abstraction. What did Piaget say is the catalyst for making this progress take place? He wrote that hands-on experience should come before verbal or abstract reasoning: "logic does not arise out of language but from a deeper source and this is to be found in the general coordination of *actions*" (Piaget, ***Mathematical Education***, 1972/1977, p. 726). Concrete activity with objects, particularly for young students, is considered to be an essential means of comprehending arithmetic and geometry.

Besides doing physical activities, the other essential key is that the objects should correspond with the child's present understanding of the world. By being able to link the new learning to existing internal structures, the child's mental structure (schema, mental model, framework) will grow. Piaget said that "the representations of *models* used should correspond to the natural logic of the levels of the pupils in question, and formalization should be kept for a later moment as a type of systematisation of the notions already acquired. This certainly means the use of intuition before axiomatisation..." (Piaget, 1972/1977, p. 732).

Computers and Education

The Computer Culture

Andrea diSessa, of MIT's Artificial Intelligence Lab, suggested that:

> "being able to choose and walk a straight path across the room is every bit as much knowledge (albeit knowledge within process) about geometry as Euclid's axioms... axiomatics cannot engage the more general style by which people quickly and effectively learn about the world. People are more fundamentally model builders than they are formal system builders." (diSessa, 1978, p. 17)

This modeling tradition has been carried into the computer world by Seymour Papert (inventor of Logo), who also worked in MIT's AI Lab. His idea of restricted environments (or microworlds) enables a child to concentrate on a few perceptual cues as well as to creatively participate in its formation: "*the child programs the computer* and, in doing so, both acquires a sense of mastery over a piece of the most modern and powerful technology and establishes an intimate contact with some of the deepest ideas from science, from mathematics, and from the art of intellectual model building" (Papert, 1980, p. 5).

He described deficiencies in the normal way of teaching math and physics: dissociated patterns are taught as prerequisites to learning the "interesting" material beyond what the student's motivation will tolerate. Mathophobia and boredom often result. Instead of this, he suggested a new instructional method: "a computer-based interactive learning environment where the prerequisites are built into the system and where learners can become the active, constructing architects of their own learning" (Papert, p. 122). He called it "a Piagetian learning path" and hoped to enable children to enter a "Mathland" culture, where they could more easily learn math, just as one could learn French more easily in France.

If this all sounds like a fantasy to you, consider the fact that in 1993, thirteen years after publishing ***Mindstorms***, Papert was awarded the Software Publishers Association: Lifetime Achievement Award and is currently the director of MIT's Media Lab. The SPA said:

> "More than any other individual, Seymour Papert has been responsible for transforming the way education views the role of computer in schools. Papert showed that computers are ideal tools to promote creativity, critical thinking, and problem solving" (***Electronic Learning***, 1993, pp. 8-9).

Public opinion has followed polar extremes, from promoting his new computer language as educational *magic* to prominent rebuttals of his ideas, such as Jan Hawkins of Bank Street College stating that Logo has promised more than it can deliver (Hassett, 1984). Papert answered, in 1985: "Logo is not a person; it neither promises nor delivers. It's a medium of expression...I'm struck by the variety of forms it takes...in different cultural, social, and educational settings...there is, of course, no 'right way' of using Logo" (Papert, 1985, p. 3).

This study describes an educational game that uses computers in many different ways, including presentation of multiple choice tests (and automatic recordkeeping), an intelligent tutoring system, and colorful graphics. Some of the modules will give more or less options to the student in color; text or animation; control of construction. Some of these choices will be discussed in more detail below.

Conversation with the Computer

There are two basic branches of ITS interfaces: "first-person" and "second-person" interfaces (Miller, 1988). A first-person interface allows the person to have "direct"[1] interaction with the computer: often just clicking on a Macintosh icon will result in a dynamic, graphical change. On the other hand, second-person interfaces act through an second layer, frequently with the use of a programming language.

The computer game mentioned here would hopefully include both options. Generally, control of the interface should be accessible to even young children (first-person). However, a few, more sophisticated sections would allow use of programming commands (second-person). Examples of the second-person interface would be (1) the Freeway module, which lets the user build roads in small sections--then travel on completed pathways; or (2) Open Construction, which would allow building in Logo-like 3-d. Each of type of interface has its own educational advantages.

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[1] As McTear has said in ***The Articulate Computer****, people definitely can't talk to computers, and computers only *appear* to talk to people. What's even worse, merely a small detail in error can cause a computer to act as though it is dead or catatonic!

Computers in Math Education

In ***An Agenda for Action: Recommendations for School Mathematics of the 1980s***, #3, the National Council of Teachers of Mathematics stated emphatically that: "Mathematics programs must take full advantage of the power of calculators and computers at all grade levels" (inner cover, NCTM, 1981). They included their position statement on Computers in the Classroom (from October, 1980), in which they listed some mathematical uses. Many of their general aims are the same as this author's:

> Mathematics programs designed to take full advantage of the multidisciplinary potential of computers should include the following:
> - Problem solving
> - Simulations[2] that give the opportunity to practice decision making
> - Lessons that introduce and develop concepts...
> - Simulations that replace dangerous, expensive, or technically difficult laboratory work
> - Programming...
> - Functions that improve the evaluation process
>
> Curriculum materials should be developed that capitalize on unique characteristics of the computer. Such materials should provide instructional experiences heretofore impossible as well as imaginative approaches to existing components of the curriculum.
>
> (NCMT, 1981, inner cover)

There are now many arithmetic-tables software packages available for the math classroom. Many of them are basically *drill and practice,* encased in colorful graphic games. Some of these are extremely useful and very engaging, but are not what we are considering here. In describing the difference between drill and practice and simulations (here, equivalent to microworlds), Nils

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[2]*Simulation*: Programs that attempt to represent key aspects of some environment within which the user will experience the necessity to make decisions and will be informed of the results of those decisions *without experiencing the real consequences of possible misjudgements*. The time required to develop and use simulations with high fidelity is justified in situations where actual experience is ruled out because of extreme expense, safety considerations, or the time required for the actual experience. Siumulations include problem solving tasks (e.g., negotiation of a bank loan, diagnosis of illnesses or equipment failures, genetic experiments, testing theoretical models), procedural tasks (e.g., acid titration, blasting, the breeding of organisms), and performances (e.g., control of water pollution) (NCTM, 1981, p. 13)

Peterson (1984, p. 287), wrote:

> In drill and practice, the computer attempts to program the student with certain facts. The student is a passive learner. Simulations, however are active learning environments. They provide a world for the learner to explore...In addition to facts, simulations teach the skills of the explorer: scientific method, debugging, and hierarchically organized thinking.

This and other kinds of software are more similar to what FunFunctions is designed to contain: (1) *intelligent tutoring system:* inside, an intricate hierarchy, based on domain content, that is used to guide tutor's decisions; (2) *spatial abilities mastery:* 3-d transformations, rotations, and orientation; (3) *dynamic computer graphics:* interactive control by the student; (4) *accurate coordinate system in 3-d:* architectural software; (5) *simulations:* active learning environments (as in microworlds).

Examples of these are:

(1) ITS: CMU mathematics tutors;

(2) spatial mastery: Sachter's J3D (Sachter, 1990);

(3) and (4) student may guide graphing of functions: Mathematica, MathCad, and Maple; 3-d accuracy: virtual reality and AutoCad;

(5) simulations: CMU's electronic shuffleboard (for teaching physics); ChemTool (only allows certain chemical bonds).

There is a large amount of overlap between these categories. A few generate the scientific visualization mentioned above. Some are more accessible to a student than others, in being immediately interactive and front-end usable. The learning curves are very long in much of the 3-d modeling software (the last two examples). Some are mostly in text (1), while others have beautiful graphics. Often, this curve (and amount of computer memory needed) increases when there is a high amount of aesthetic appeal and the user can construct freely. For example, architectural software encompasses accuracy, beauty, and freedom, but takes years to completely master and is time-consuming to use in a detailed project.

If FunFunctions grows as it should (matures properly?), it will integrate all these features into an easily used interface. This is one of the major changes that should be added to 3-d modeling software: child-usability. There are already a few attempts at this, but they usually lack accuracy, preserving perspective without the mathematical accuracy of architectural software.

Also, once the interface is in place between all the modules and tutorial levels, an easily usable front-end is in place, and testing has been done, the whole gameboard could be separated from content-knowledge, leaving a shell that could be used by others. This would be similar to the way that MYCIN, an diagnostic medical system, was used to generate a shell that could be used for other instructional purposes: MYCIN was the original software, which contained a large knowledge-base of medical information (an expert's facts), as well as a way to search and make decisions (inference engine). Then, the knowledge was separated out, leaving an "empty" (thus *E*MYCIN) *shell* that others could use (Knox-Quinn, 1988).

Existing and evolving intelligent tutors at CMU (in math, economics, and programming) have been encouraging role models, for me: (1) they combine at least two *linked representations* (hypotheses with geometric drawings; text, equations, to graphs; etc.) and (2) incorporate internal production systems[3] based on the subject being taught and the present state of the student's *understanding*.

On the other hand, *constructive* geometric software (such as The Geometric Supposer, The Geometer's SketchPad, and Cabri Géometrè) gives an example of how to create environments with *constraints.* They are meant to lead the student toward a better understanding of the rules that set external limits on their activities. This would be *inductive reasoning*, defined in ***Discovering Geometry*** as: "the process of observing data, recognizing patterns, and making generalizations from your observations" (Serra, 1989, p. 39). Instead of being a concrete model of algebraic concepts, dynamic models capture better the general understanding of variables (Kaput, p. 529). Traditional "literals" show an instantiation frozen in time, not the functional

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[3] *Production systems* -- are manufactured by breaking down procedures into one-step choices. As each operation is encountered in the program, the computer is preprogrammed to act upon it or not, depending upon the present conditions (state). In humans doing algebra, for instance, this might be thought of as the numerous rules used (almost unconsciously) to factor a number (Simon, 1988).

concept of variables changing. Another beneficial use of computers in mathematics education was mentioned above, in "Educational philosophies," computers as *manipulatives.* Ohlsson (1987) wrote that using solid objects and pictures to *illustrate* arithmetical concepts may not be as helpful as was assumed, due to (1) not being similar enough (not isomorphic), (2) the knowledge will transfer, and (3) it may be as hard to learn as the original was. However, he wrote that computers could aid this illustration by *yoking* it to mathematical concepts. For instance, a child could be given control of how many objects to generate (on the computer screen). The computer could automatically show the mathematical equation for this: "When an illustration world is defined on a computer, the generative character of the definition translates into exploratory freedom for the learner. The learner can literally discover how mathematics describes the world by creating any arbitrary configuration...and then have the computer tell him what the correct mathematical description is" (Ohlsson, 1987, p. 341).

A related use is *visualization:*

> The first step in data analysis is the visual display of data to search for hidden patterns. Graphs of various types provide visual display of relations and functions...For centuries artists and map makers have used geometric devices such as projection to represent three-dimensional scenes on a two-dimensional canvas or sheet of paper. Now *computer graphics* automate these processes and let us explore as well...Learning to visualize mathematical patterns enlists the gift of sight as an invaluable ally in mathematical education. (Steen, 1990)

Computer Games

There are many reasons to use interactive computer games: (1) a "microworld" can be organized to create a parallel universe where there is a certain sense of reality while all the nonessential factors are dropped out; (2) motivation is raised by the challenge to conquer the game (see Malone, page 25); and (3) the child receives individual feedback to his responses (cued to the recorded history of correct answers and errors, and constrained by rules of the microworld). Also, recent technical advances have made more realistic and accurate computer graphics and animations superior to representations in other modes.

A different author from MIT's Artificial Intelligence Labratory, Brian Carr (1977) described why games, and specifically ***computer games*** are very appropriate as an educational tool. He said that games can be fun and still teach valuable skills, "an ideal teaching instrument." Yet, he wrote, one must be careful in how one constructs such a computer game. His was an early effort at creating an intelligent system that would make use of the program's ubiquitous presence and *also know when to step in and offer advice*:

> However, students can reach a plateau in their game playing and cease to try new strategies. When this occurs, the learning process halts. A solution to this problem is to encourage improved game strategies. Unfortunately, the cost of providing human teachers to watch the game (and offer suggestions) is prohibitive. However, it is possible to use a computer to monitor the progress of the game and to offer suggestions when warranted. With this goal in mind, a program was written to serve as just such an advisor for a computer game (Carr, 1977, p. 3).

This is a very important, and constant issue to consider. Throughout each game module in FunFunctions, content and timing of advice must be preprogrammed. Much of it depends upon the present concept being taught. Sometimes it is self-evident how to give advice. At other times, the only answer is to automatically branch the student to another part of the game set.

Game Passageways

Concept sequence. The games in FunFunctions are to be linked together in Nintendo-like fashion, but instead of secret passageways, real concepts must be mastered in order to pass on to the next level. Eventually, the whole game should be linked together, mostly in an mathematically understandable way (as in the Van Hiele levels). At this time, however, the games were presented in separate modules, with only two passageways (between Perspectives and Cubes; and between Mountain and Freeway) in place.

Random. However, planned pathways may also be supplemented by unexpected passageways, by the player's choices or the computer's random generation. This would help to increase interest (see the following section) without harm, if the important sequential structures are in place. Singley, in ***The Transfer of Cognitive Skill*** (1989, p. 19), stated that:

> "Many researchers have failed to find any effect for principled curriculum design...Notable among these failed attempts were the *scramble* studies, where carefully designed programs of instruction were pitted against versions whose frames had been randomly ordered...One moderately successful study was done by Buckland (1968), who found that less capable students were hurt by the scrambled curriculum but more capable students were not."

Perhaps many of the connections could be set "randomly," left open to the player's choices, or set in alternating ways to record the effects of learning math in different sequential patterns.

Palette. Another difference between this version and the later, more complete one, would be that the the later one might have replaceable modules; mastering one area could give one the choice of replacing it with more advanced areas (calculus, etc.). Also, special interest areas could be added, depending upon age, interest, and special needs; similar to an artist's choice of palette colors, which changes from day to day, depending upon the individual picture to be painted. This would be supervised by the teacher, who would choose which modules to display, influenced by the student's suggestions.

Multiple modes. In fact, one of the primary envisioned purposes of this game was to use it with students who have trouble with verbal communications, but have superior spatial skills (as in students who have learning disabilities (verbal), but high video game mastery (spatial)). Through this game, with its requirement of passing parallel levels of the same questions (in Mountain Perspective's: Text, Diagram, and Animation tracks, for example), multiple representation would help to reinforce concept, while linking stronger areas to weaker ones, in a supportive role.

A person with cerebral palsy wrote about how computers were used to release his "trapped intelligence." In this case, a computer's keyboard (tactile mode of communication) was used to overcome a deficiency in verbalization:

> Trapped intelligence is a phrase which is used to describe people who have normal or above normal intelligence but are non-verbal or slow talking and *society assumes that these people are stupid*...For the first nine years of my schooling,...nobody knew what my potenial [sic]was or would be...[after working with a computer language, we found that] a scientific break-through had taken place...new avenues of communication and education have been opened. (Micheal Murphy, in Valente, 1983, pp. 129-130)

The opening of new modes of communication has also been used in mathematics education by Sylvia Weir and E. Paul Goldenberg. For instance, one student was able to divide the perimeter of a regular polygon into the correct number of units when it was presented as a problem on the computer screen (visually, with "hands-on" exploration), although previously unable to solve it (Weir, 1987; Goldenberg, Russell, & Carter, 1984).

How to Make an Educational Game Interesting

Thomas Malone's theories explain the characteristics that make an educational game captivating. As he said, his "studies focus on what makes the games fun, not on what makes them educational" (Malone, 1981, p. 333). He organized the most important characteristics into three main categories, which each include several factors: Challenge, Fantasy, and Curiosity.

Under **Challenge**, Malone included Goals (personally meaningful, obvious, as performance feedback); Uncertain outcome (variable levels of difficulty and goals, hidden information, and randomness); Toys vs. tools, and Self-esteem. He stated that toys and tools are often at opposite ends of the spectrum, so far as challenge goes: "the tool itself should be reliable, efficient, and usually 'invisible'. In a sense, a good game is intentionally made difficult to play to increase its challenge, but a tool should be made as easy as possible to use" (Malone, p. 359). By self-esteem, he meant that the level of difficulty should be set so as to encourage a sense of success: have variable settings that can be adjusted appropriately and give feedback in a nonthreatening way. Of course, this must be done in a balanced way. Otherwise, the game would be unchallenging, and thus boring.

Fantasies are either Intrinsic[4] or Extrinsic, and have both Cognitive and Emotional effects. He said that in most intrinsic fantasies, "problems are presented in terms of the elements of the fantasy world, and players receive a natural kind of constructive feedback...In general, intrinsic fantasies are both (a) more interesting and (b) more instructional than extrinsic fantasies" (p. 361). Cognitive effects include the aid that old knowledge gives when used as a metaphor in modeling new concepts: understanding, restructuring, and memorizing the new ideas would be improved. Emotionally, different fantasies are needed for different peope: fulfilling varying emotional needs, social roles they see themselves in, and, perhaps, there are rewards to presenting them in the best learning modality for each separate person. Malone suggested giving the player a choice of imaginary participants and problem fantasies.

[4]"Intrinsic" - in this sense, means that *playing* the game is its own reward.

Curiosity, the third category, means that the designer should search for an optimal level of complexity. This includes Sensory curiosity (audio and visual effects) and Cognitive curiosity (well-formulated knowledge structures and informative feedback) at a level that is "novel and surprising, but not completely incomprehensible...There is no reason why educational environments have to be impoverished sensory environments" (Malone, p. 362-3). Malone said that cognitive curiosity takes place when the "knowledge structures" (or story plots) are complete, consistent, and parsimonious. He gave the example of the strong motivation we have for finishing the last chapter of a murder mystery. The other important aspect of cognitive curiosity is informative feedback. In order to fulfill this, the feedback should be surprising and constructive (showing one *how* to change their mental model to make it more complete and accurate).

Other structural features (mentioned by Malone) that encourage motivation are: the use of multiple perspectives, lack of scoring on new ideas, ability to make inferences about hidden information, personalization, and choice.

Malone (p. 356) tied his three main categories in with Piaget's ideas: "he [Piaget] claims that people are driven by a will to mastery (challenge) to seek optimally informative environments (curiosity) which they assimilate, in part, using schemas from other contexts (fantasy)." However, as Malone mentioned, these are difficult to interpret educationally. Malone's ideas have given us a much more detailed level of resolution, which is helpful when assembling educational courseware.

Application

How are Malone's ideas (Fantasy, Challenge and Curiosity) applied to the design of FunFunctions?

Levels. Some of game modules can only be entered through successfully completing a lower level. Also, the teacher would be able to control availability of certain modules (for instance, a third-grader would not see an icon for a game using calculus). Other games would be removed if they were too easy for the user. Invisible record-keeping should be used to decide whether enough component subgoals had been reached to allow the student to enter any of a number of possible game modules, in a branching manner.

Student Modelling. This would have to be based on student modelling (or modeling) procedures, which show the difference between "expert" knowledge and the student's present state of knowledge. This may be done by *overlay*, in which the student diagnostic model is assumed to contain (only) part of the expert's knowledge; or with a *bug library (or bug parts)*, which then adds on student misconceptions. Interactive behavior, determined by the student's accurate understanding or false knowledge, may be diagnosed by *the final result* (the answers chosen), in tiny steps which are then matched to an overall model (*model tracing)*, or by dynamic generation, *finding a path* that will lead from the student's activities to the nearest model (VanLehn, 1988; Ohlsson, 1992*). Then the tutor will have to "decide" what to do next: fill in conceptual gaps,advance the student to a higher level, or correct misconceptions.

Unexpected pathways. Unexpected pathways and blockades between different game modules could also be used. Some of these would be based on logical sequences (see the following section), while others would change from time to time (according to student choices, computer random generation, or teacher management).

Order of presentation

Although much learning can be "absorbed" spontaneously by experiencing "the computer culture", a certain amount of domain knowledge is sequential (depending upon the subject area). For instance, in arithmetic, addition is prerequisite to an understanding of multiplication. Not everything needs to be in a set order (and will indeed add to game interest, by being unexpected; see Malone's theories, above) but some subjects are constrained, when they contain basic areas that will later be subcomponents of higher level knowledge. Gagné, working with other educational psychologists, decided that curriculum design should be influenced by *learning hierarchies.* In some cases, learning could transfer almost automatically, such as in the learning of computer programming languages, but at other times, subordinate skills (or learning sets) would have to be mastered before "higher-order skill," or the aim of the instruction, would be learned.

Deciding how to structure these learning hierarchies of component learning sets should be an integral part of instructional design. Problems with this approach are two-fold: in finding the proper decomposition of skills (task analysis) and in matching them with the individual student (such as knowing when to allow the more capable students to skip certain component parts).

Other theories about sequences include Bruner's *spiral curriculum,* where ideas are reintroduced at more advanced levels, each time building in detail; Ausubel's *progressive differentiation*, where general statements prepare the student for complex topics; and Reigeluth's ***elaboration theory,*** where amount of abstraction, or resolution, zooms back and forth, to give the student a bigger picture of where the knowledge is fitting in (Singley, p. 16-20). While teaching, an ideal tutor would decide where each student, at each moment, fits in to the overlay model (of expert's knowledge) and bug library (of misconceptions), before choosing what to teach next. "In general, the next topic should be chosen from those subject-matter units which are not yet mastered, but which have all their prerequisites ticked off as known" (Ohlsson, 1992, p. 208).

Math sequences

Piaget worked out an intricate sequence of mathematically-based learning after creating a meticulous study of learning tasks. He wrote that topological learning precedes projective, which is then followed by Cartesian coordinates. It is interesting that this is the reverse order of how math coursework is usually taught: topology *follows* many years of algebra and other courses (which all include x-, y- coordinates); and 3-dimensional figures are taught *after* plane geometry (except for a smattering of decorative polyhedrons). Meanwhile, perspective-drawing, related to the projective viewpoint, is usually taught to artists and vocational tech students, not as a math course at all.

"Historically," said Piaget, "these intuitions appeared in Euclidean geometry, the structures of projective geometry were not discovered until much later and topology only in the nineteenth century," while children of three and four can distinguish topological concepts (open and closed areas, for instance) progressing to:"later and simultaneously, projective notions (with verification by 'taking aim' or 'sighting') and Euclidean notions according to a process which is nearer psychological theory than history" (Piaget, 1972/1977, p. 729).

Yakimanskaya et al (1991, p. 202) emphatically agree with Piaget's analysis, in place of the present system (both in his country and America), saying: "The current mathematics curriculum actually realizes a reverse pattern," ignoring "psychological laws." Yakimanskaya et al write that, among other things, the students initial grasp of 3-d space is inhibited by the constant use of merely 2-d representations in geometry: "The rich store of experience, accumulated from manipulating real (solid) objects [in primary school] seems to be suspended once they start plane geometry, since the content and logic of this subject entails the exclusive use of two-dimensional representations...Moreover, the constant use of two-dimensional representations leads to over-attachment to a fixed observation point" (Yakimanskaya et al, p. 134).

There have been some additions to Piaget's ideas in the decades since his books were first published in English. Not everyone agrees with him. However, one currently used mathematics education structure that remains related to Piaget's theories is the Van Hieles' hierarchy. The following Van Hiele levels are adapted from Clements & Battista (1992, p 431):

(Level 1) *Information.* Students learn vocabulary, and geometric shapes. The teacher discusses them, and listens to the child, in guiding his/her teaching.

(Level 2) *Guided orientation.* Students explore geometric objects (for example, folding, measuring) under the direction of the teacher.

(Level 3) *Explicitation.* The children become aware of their geometric conceptualizations, and begin to learn some of the traditional language for this subject. The teacher discusses geometric ideas and patterns with the class in "their language," then introduces the traditional math terminology.

(Level 4) *Free orientation.* The students learn to structure and use the geometric objects and ideas they have learned about earlier. The teacher selects appropriate geometric problems (which have more than one solution path) and introduces relevant terms and tools.

(Level 5) *Integration.* The students build a coherent network out of what they have learned so far. Eventually, the teacher leads them towards the structure of formal mathematics.

Textbooks

Addison-Wesley Publishing Co. overtly makes use of this hierarchy in their 1992 textbook, ***Informal Geometry***. They also have a checklist of four items that link this book to the ***Standards*** and also this study's computer game (p. T1):

- Intuitive learning through hands-on activities and *visualization* exercises;
- Connections to algebra and the *real world* with applications;
- *Discussion exercises* encourage communication and cooperation;
- *Technology* is integrated into this course.

Another book, ***Discovering Geometry: An Inductive Approach***, by Michael Serra (1989), has similar qualities. It is perhaps no coincidence that it has been used in conjunction with The Geometry Proof Tutor and Angle, intelligent tutoring systems at CMU. Only a third mainstream textbook making obvious use of spatial visualization has crossed the author's path: ***Spatial Visualization***, part of the ***MGMP, Middle Grades Math Project***, also published by Addison-Wesley (1986). Other activity books in this vein are usually published as supplements, not part of the main curriculum.

Despite encouragement from the Van Hieles, Piaget, and the ***Standards***, visualization has not *yet* made great headway in the standard math curriculum. There is a need for spatially-oriented curriculum, and especially software, which enhances the inductive approach.

Spatial Realities

Three-dimensional models

So far, most of the existing microworlds have been on a flat plane. That is one of the main differences between FunFunctions and the other microworlds. The use of a three-dimensional model would be even more realistic (which is especially important to the younger child, according to Burton. See "Fidelity and Mental Models" above). Also, it could be used to help them improve their spatial understanding, and thus, their math skills.

Spatial Understanding and Math

NCTM's ***Curriculum and Evaluation Standards for School Mathematics*** (1989, p. 48), includes spatial understanding as a focus issue: "Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world." Jean Shaw wrote in the ***Arithmetic Teacher*** that, "Without spatial sense and the vocabulary to describe relationships, we could not communicate about positions or the relationships of two or more objects. We could not give and receive directions for finding locations or completing simple tasks...We would be hampered in our abilities to analyze figures and the relationships of their parts" (Shaw, 1990, p. 4).

Career success in many disciplines may be related to high spatial abilities[1]: Ben-Chaim, Lappan, and Houang (1988, p. 52) stated that investigation of spatial perception is to be valued "for its relationship to most technical-scientific occupations and especially to the study of mathematics, science, art, and engineering". Relevant quotes abound about the value of this "spatial understanding."

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[1]Some authors have distinguished between *skills* and *abilities*. The latter are what we are born with and the former are the areas that can be developed. In the ***Webster's New Twentieth Century Dictionary, 2nd edn.*** (1979), *ability* is defined as "power to do (something physical or mental)," while *skill* is derived from the Old Norse word for discernment and knowledge. In this paper, we are concentrating more upon the areas that can be improved by teaching knowledge and discernment, the *skills*.

The two main branches of spatial abilities/skills are *spatial visualization* and *spatial orientation*. Spatial visualization means mentally moving or transforming an imagined object, either by turning it or rearranging it. The other branch, spatial orientation refers to the understanding of visual patterns, coordinate systems, and ability to change one's perspective.

However, since spatial skills are essentially *non-verbal*, it is more difficult to discuss or publish detailed reports in this subject area than about areas that are more easily articulated. Defining of terms is a problem, in itself. Metzler and Shepard (1974) theorized that the study of 3-D rotations might *seem* like an obscure area because of verbal inaccessability,while authentically being very important:

> ...the possibility should be considered that the long-standing preoccupation of psychologists with exclusively verbal processes in learning, memory, problem solving, and the control of behavior generally may be a reflection more of the relative accessibility of verbal processes than of the preeminent role that verbal processes play in human thought...It is hardly surprising that, during its long submission to the strictures of extreme behaviorism, psychology found little room for even the term "mental image." (p. 70-71)

However, since that time, partly due to their efforts and extensive investigations, mental models, imagery and spatial abilities have become much-more discussed topics in the last two decades.

Hard to Define

Historically, researchers studying general intelligence factored out at least one spatial variable (distinct from numerical and verbal factors) at the beginning of this century. Yet, after many decades, ambiguities remain in this area. And although it has been found that there is a high correlation between certain spatial skills and math achievement, the exact nature of this relationship is not clearly defined:

> Researchers have also investigated various skills believed to contribute to mathematics learning and have found spatial skills to be related to mathematics achievement. Using various techniques, studies have demonstrated that many spatial skills are positively correlated with a wide variety of mathematics tasks. Yet the precise nature of spatial skills and the manner in which they contribute to or predict performance in mathematics... is largely unknown. (Tartre, 1990, pp. 27-8).

One of the major findings of spatially-related investigations was the gender-linked correlation with math achievement. It appears to be one of the main factors that covaries with math ability, differentiating males from females, while staying proportional with math achievement (Leder & Fennema, 1990):

> Spatial ability is one of the factors most consistently linked to gender differences in mathematics achievement, one dimension of educational outcome. The conflicting findings--excerbated by definitional confusion--that have characterized much of the research suggest the need for carefully structured and clearly described further study.

Some studies have found great sex-linked differences, while others have considered them insignificant. In one school, when certain spatial skills were factored out, gender differences became very small: "Using scores on spatial visualization as a covariate, the difference between the sexes in mathematics achievement became nonsignificant" (Fennema & Sherman, 1977, p. 64). Further investigation into these relationships would be of academic value, both to scholars and to teachers.

Improving Spatial Skills

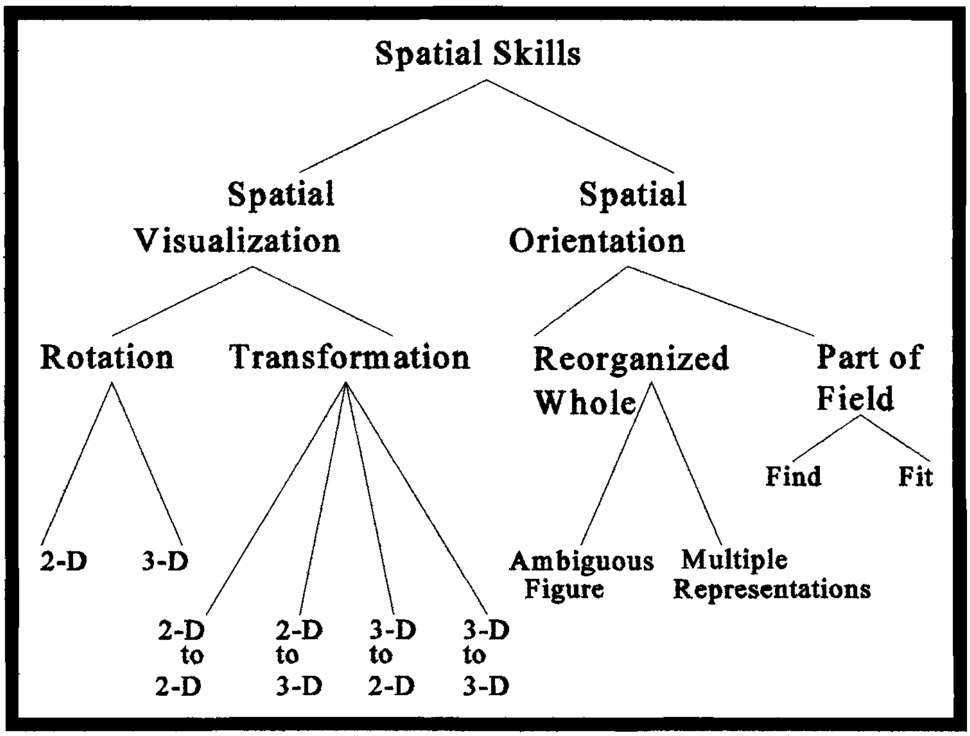
Although the limited scope of this research (small n value) and the early stage of the software's development prevents any decisive conclusions about either the development of spatial skills or correlation with math skills, this exploratory study is expected to reinforce past research findings. Most important of all, it should be helpful in guiding the development of this supplemental instructional program, which aims to give spatial training. At least this is one thing that every researcher (who writes about it) agrees with: the equalized benefits of spatial training.

One such study was completed by Ben-Chaim, Lappan, and Houang (1988). They found sex differences in spatial abilities in sixth, seventh, and eighth grade students, but not in fifth grade students. After this, they conducted a much larger study involving 1000 students. This time, they found: "The most important result of this investigation was that after the instruction intervention, *middle school students regardless of sex gained significantly from the training program in spatial visualization tasks.* The students made across-the-board gains in posttest item types" (p. 66).

Specific Spatial Skills

A central problem has been to define *spatial skills*. As Lindsay Tartre says, "Each researcher seems to have his or her own definition of 'spatial skill' " (Tartre, p. 28). No wonder scholarly progress in this area has been sparse: contradictory, overlapping, and/or vague terminology has often confounded clarification of research findings. Fortunately, Lindsay Tartre has distilled the branches and redefinitions of these skills into an easy-to-read tree (see Figure 2). For a very helpful sample of related test questions, also see the meticulous details in the book ***Math and Gender*** (1990).

Figure 2: Specific Spatial Skills



From "Spatial Skills, Gender, and Mathematics" by L. Tarte, 1990. In E. Fennema & G. Leder (Eds.), ***Mathematics and Gender*** (p. 30). New York: Teacher's College, Columbia University.

Explanation of Research Correlations

Some researchers have attempted to explain the correlation between spatial and math skills:

> Many hypotheses have been suggested to explain how and why spatial skills relate to mathematics performance. Smith (1964) proposed that "spatial abilities may be involved in the perception and assimilation of patterns, either in the structure of geometric figures or in the more general structure of mathematical symbolism..."(p. 125). Schonberger (1980) stated that "the use of charts, diagrams, and graphs in all branches of mathematics argues for the logic of this connection between `spatial ability' and mathematics" (p. 189) (Tartre, pp. 45-47).

Other authors have attempted to explain how both spatial and math skills are correlated with gender differences. The appearance of differences has been credited to many things: different perception of space, eyetracking, testing, genetics, strategies, societal pressures to do poorly in math, physical activities (such as the fact that boys tend to play in activities that move through space), previous classes(either spatially or math related), or interests. This range is almost equivalent to the nature or nurture controversy, and just about as unsolvable.

However, once again, there is a retreat from this argument to the much more constructive concept of training: whatever the differences are due to, they have been able to be improved, which is supported by experimental results (see above). But how should one go about this training? Many of the reasons for choosing computer graphics have been listed in the "Computers and Education" section. However, this is not the sole means of teaching. An integrated approach (with a person teaching, solid models, pictures, and explanation of concepts and goals) would be the more beneficial. The computer section of the instruction would be only a supplement, perhaps lasting three weeks. Of course, the concepts would have to be carried throughout the semester by the teacher, in lecture and class activities.

Why Use Computer Graphics to Teach Spatial Concepts?

There are several facets to this answer: the "intrinsic"[2] relationship between 3-D graphics and the X-, Y-, Z- coordinates in reality; the close to real-time response time; the ability to interweave far-branching questions; the ability to respond "intelligently" to the child in the absence of human tutors; and the child's attraction for anything on the screen. Also, it is more convenient for the teacher, who now only has to draw a 3-d model once (not repeatedly for every perspective) and has the added benefit of multiple views or examples, thus making generalization and comprehension of the drawing more accessible to the children.

Judy Sachter worked with 10 and 11 year-old children (the same age as in this study). She found that the children enjoyed 3-d computer graphics and "created wonderful work with it." She wrote that it would be of great educational value for children to have access to it, despite its difficulty for them to use it (due to the interfaces):

> Presently 3-D computer graphics is being used a great deal to study various scientific phenomena through simulations, scientific visualization, dynamic systems, and in the design process in many fields (e.g., architecture, engineering, automobile design, etc.). I believe that all of these fields that currently use the medium could offer a great deal to the *educational field* if children had access to it to play with science and to design artifacts. It is in the power of computer simulations that this benefit becomes magnified. Not only is 3-D computer graphics a place where children can explore space, but it could also be a virtual world, where one can go from the microcosm to the macrocosm, from science to art, and back again. 3-D computer graphics has always been an interdisciplinary field of computer science, and it has moved into the world as an environment that both artists and scientists have appropriated as their medium of choice. (Sachter, 1990, p. 250)

The invention of an easily child-controllable interface would be of benefit to many people, including Judy Sachter, who has already imagined and done some research on this possibility.

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[2] "Intrinsic" representation - means that there is a structural equivalence between the data and the model. For instance, length of three sticks could be shown by a barchart in the same proportion. If an extrinsic procedure were used instead, such as making each point (representing a stick) extend an arrow to every point shorter than it, this would be *much* harder to interpret visually. The 'natural' correspondence is much easier to understand (Palmer, 1978).

Moving through virtual space, immersed in a head-set, is a much more intrinsic representation of physical space than a flat outline map would be, for example.

From Everyday Reality to Math Notation

In one section of the game set (Mountain Perspective) questions were presented in sequential order, because I wanted to gradually draw the students from their everyday spatial perception of the world (via intermediate representations) into an understanding of standard math notation. In this situation, the questions ranged from looking at mountains from different angles to counting the facets on 3-d wire-frame polygons copied from a textbook's geometric notations.

This is meant to create an eye-opening effect that would hopefully stay with the student from then on. An appropriate quote is by Richard Lesh (1990): "Once it became natural to view the world through Descartes' three-dimensional glasses, people tend to have difficulty remembering what it was like before this conceptual framework was constructed" (p. 92). The purpose would be to change the way the student views math wireframes, *not* to be a boring, repetitive drill. Once the new "term" is learned, there would be no *necessity* to eye-opening exercises. However, repeated practice of spatial skills, as mentioned above, is helpful as found consistently by many researchers (including Ben-Chaim, Lappan, & Houang, 1988; Pallrand & Seeber, 1984).

Concrete-Abstract

In effect, the software should help children to cross the chasm between concrete thinking to formal operations (in the Piagetian sense). For this reason and other reasons (such as the existence of 10-and 11-year old sons!), the author chose to begin work with fifth-graders--at the projected Piagetian awakening/practice time for this new world.

By creating microworlds that are constrained by only a few rules at a time, as opposed to the ill-defined swirl of factors that one is faced with in reality, the teacher can choose to emphasize certain themes. In reality, thousands of forces affect our everyday life (according to chaos theory: a butterfly's flapping it's wings in Japan may affecting the weather here). In a microworld, only a limited number of factors are chosen. For this reason, they stand out more easily, while nonessential themes drop out.

In this way, the novice can be trained to think like an expert: ways of looking at a subject can be funneled into a canonical form (a shortcut method of abstract thinking, used by the experts). For example, the physics major is trained to think in terms of vectors, a force that is seen as an almost physically-sensed microworld-type force--but "physical" in the mind's eye, not reality. In some ways, this is restrictive to the other creative ways that this budding scientist might think. But it is like another language he must learn. And after the new scientist realizes that such microworlds *can* be constructed, he may discover his own way of seeing things. Instead of being merely knowledge-based (as in learning arithmetic facts), knowledge-based learning (focused on understanding) should be emphasized. Karl Popper remarked:

> "...What I suggest is that we can grasp a theory only by trying to reinvent it or to reconstruct it, and by trying out, with the help of our imagination, all the consequences of the theory which seem to us to be interesting and important...One could say that the process of understanding and the process of the actual production or discovery of...[theories, etc.] are very much alike. Both are making and matching processes...The matching aspect is that it has to fit into a framework" (Popper & Eccles, 1977)

<u>Overcoming Gaps</u>

Another application for this concrete-abstract bridge would be in special education. Especially students with learning disabilities could benefit from this, whenever they have high spatial and low verbal abilities. Then, their spatial abilities could be capitalized on in a way that (1) is often commercially unrepresented (for example, only one well-known company at a recent math convention had a unit on spatial abilities; others, infrequently, included it in a half-page box); and (2) could be used as a conduit to other seldom-used areas in the individual's life (for example, using a high-spatial video game to lead the student to a more verbal area, especially if it is a learning disability area).

This is a frequent occurrence in students with learning disabilities: low verbal and high spatial abilities. It was also evident with one of this study's students, who rated highly in spatial tests and puzzles, while lower verbal ability correlated with low math scores. Improving spatial abilities should improve his math scores, as an alternate route of communicating math concepts is found. Also see "Multiple modes" under "Game Passageways."

For students with cultural, gender, or physically-linked gaps in their math skills, both spatial (see above) and computer training would be helpful. Both the *ability to visualize models* and familiarity with *computers* has been linked to success in math and science, and thus careers. Smith (1964), with the U.S. Employment Service, listed spatial ability achievement as a predictor of career success in mathematics, engineering, and many scientific categories; Pallrand & Seeber, (1984), linked spatial ability with success in physics; and mathematics, again, Fennema & Sherman (1977).

Other studies have linked avoidance of computer use with low success rate in many math and science related careers. One example is from Badagliacco (1990, p. 53). He found that women *expected* computer-use to be very difficult, part of "the general syndrom of females' perception of mathematics and science being more (or too) difficult for women--that is, the myth that women may be unable to achieve competence in arease that require the use of computers." Overcoming the students' "fears" of computers and math would be a great bonus in itself. Perhaps a computer game can be helpful in this way, as well as the more obvious goals of teaching math concepts, notation, and spatial reasoning.

CHAPTER 3:

PROTOTYPE DESIGN AND PILOT RESULTS

Prototype Design

General Overview of Games

The icons for a whole series of games is presented in the colorful, full-page drawing. A sketchy outline of each game's storyline and outward appearance was initially mapped (elsewhere, in notebook form). Whenever relevant software or images was glimpsed, they were sought out and included in the notebook. Sometimes, this involved seeking out interviews with local computer scientists, architects, and authors. This included visiting the art and architecture departments at Kent State University. (I was already enrolled in a math and computer teacher's institute, IFSMACSE.)

As I presented my brief ideas to (sometimes) interested people, they often directed me to more people, who were usually both knowledgeable and able to give me new, fascinating perspectives on the project. Consistently, over two years of pursuing the project *in my spare time*, I felt like I was on an adventure, similar to a *constructive* murder mystery. Some of the especially helpful people along the way were: E. Paul Goldenberg, Phil Lewis, Bill Lucak, Skip Van Wyk, Carl Loeffler, Rob Woodmont, Gaea Leinhardt, Hermi Tabachneck, Mitch Nathan, Ken Koedinger, Stellan Ohlsson, Anna Blevins, and *most of all,* Dr. Nous. Also, helping me narrow my focus, were Herb Simon and Chris Schunn.

When overwhelmed by what looked like twenty years of work ahead of me, I didn't know where to start. Several people told me to start with the simplest part of my game, make a prototype, and *then* worry about the rest of it. Finally, I chose three starting points: Mountain Perspective (like Piaget's three mountains), Freeway (construct a roadway), and the 3-d data set controller that will enable accurate passage between the two and three dimensional realms.

The first module exists in a HyperCard facade (which will later either be the connecting point of many kinds of integrated software, or live only temporarily as a prototype). The last two have their plans and software set for next semester: using virtual reality (WorldToolKit for Windows) and the valuable help of one of Pitt's graduate information science students.

Specific Spatial Skills Emphasized by this Game Set

MOUNTAIN/PERSPECTIVES. This computer game is meant to sharpen the student's ability to translate diagrams in two dimensions into mental models that correspond to our everyday three-dimensional (3-D) reality. Many of the questions are viewpoint related. It includes several different facets of spatial skills (see Spatial Tree), such as 2-D to 3-D and 3-D to 2-D Transformations, 2-D and 3-D Rotations, and Multiple Representations.

MOUNTAIN/CUBES. This short, but difficult game makes use of 2-D to 3-D to 2-D Transformations, while restricting the representation of the 3-D figure. (Concrete models may used to supplement the imagination, until the child begins to understand how they are constructed.) Rotation (or Multiple Representation) of the model is also necessary.

MOUNTAIN/ANGLES is a 2-D to 2-D Transformation, where the student is building one plane surface of a mountain by combining tangram-like parts (most useful when done mentally). If the plane surfaces are combined into a whole mountain, it becomes a 2-D to 3-D Transformation.

FREEWAY. This game is meant to create an intermediate representation, a bridge between the child's sense of spatial reality and math notation. By flipping back and forth between wire-frame diagrams, created by the child, and 3-D animations (virtual reality), the student should get a better understanding of the correlation between coordinate systems and the reality they represent: a Spatial Orientation skill. (Many of the other skills are indirectly included, depending upon the individual task being pursued.)

TEACHING AIDS. Along with the games included in this prototype are a good many teaching aids. The manipulatives (Legos, wooden cubes and dice) are to be used to help the child represent the problems in a concrete way. They are another span to connect questions--concrete models--computer microworlds--imagination--abstract analysis--solutions--reality. They are a hands-on activity, which the mind remembers more easily ("Tell me, I forget. Show me, I remember. Involve me, I understand." according to a proverb.) Also they can be used to temporarily represent 2-D to 3-D and vice versa Transformations, until the students' mind begins to operate more easily in the new language of math notation.

Other teaching aids directly represent 2-D to 3-D Transformation (cutting out and assembling polyhedron models), and Multiple Representation (the colorful mountain pictures, which may also be represented in computer pictures instead of their present paper form).

Specifically, Mountain Perspective:

Mountain Perspective contains three modal representations (text, animation, and diagrams) of seven questions. The sixth question is identical in all tracks. The seventh question is a variation of one of Judy Sachter's sample questions in "Kids `n Space," (1990), an M.I.T. doctoral paper. It is the Johnson-Meade adaptation of Shepard-Metzler's cube rotations. Each *question* looks nearly identical in all tracks. However, the *answers* presented differ in the Text track, which has only *words*. Also, in the 3-d track, rough *animations* simulate movement.

The colorful mountain pictures were created by using 3-d modeling software to form one model. Different perspectives were generated by changing the viewing angle and/or the style of representation. The initial model did not have to be changed to create the many variations! These examples were generated with a quick version of Stratavision, then were roughly animated by zooming in on them (in HyperCard). Within the Mountain area, the other two choices are Angles and Cubes. Mountain Angles concentrates on angle representations. Mountain Cubes uses three-dimensional cubes to enable the children to develop their spatial abilities (listed elsewhere).

Variations: Minor differences in questions involved changing the viewing angle from left to right on the first question and using the word "helicopter" instead of "airplane" on the second question. Also, in the correct answer, location varied both within each series of seven questions and on the same question in different tracks. In the previous study, there were duplications between 3-d First and 3-d Last. One of them showed the diagrams *before* the animation (to orient the viewer), while the other alternative showed the animation first. They both used the same answer screens. In the present version, these two branches have been consolidated into one 3-d path. The alternate choice has been preserved (invisibly), possibly for comparison purposes, later. Meanwhile, each subject will only be able to take *one* of the 3-d paths.

These questions and answers are printed in miniature form in Appendix B: HyperCard stack. In a different notebook, there is a copy of an older version: Dividers mark the beginning of each section. Certain portions, for instance the floating layer of colored mountain pictures, would not print out. As a result, most of the Animations are not visible. Several pages of colorful mountain scenes are included near the end of this appendix to show the pictures that were used to create these simulations and floating pictures (which also accompanied the first five questions).

Records: Recently created records (containing a blank, covering field to shield them from students' eyes) appear to be empty cards. This protective layer has already been removed from the earlier records at the end of the stack. Choosing the "Teacher access" option would make them *all* visible, even while playing the game. Fortunately, all the data included in this analysis printed out successfully. Within the HyperCard stack, the records are created in reverse order, since they originate from a template for recordkeeping cards..

Motivation: feedback when right (sound, congratulations, and scorekeeping) culminating in an attempt for the "Hall of Fame" and the opportunity to go on to the next level; feedback when wrong (signposts and the obstruction of all forward movement).

Remediation: hints that were optional initially were automatically uncovered on later attempts. (More is needed, as the program is developed.) Also, the student was forced to see the problem in at least two different ways, because each child had to try both tracks.

For more detailed descriptions of the questions, see Appendix I: Software. In the second section, different versions of the same question are drawn on single pages. It might be helpful to refer to this appendix while reading the next paragraph.

Human tutor, students, and developers

The interweaving of design and human tutoring were apparent throughout this study. As the courseware was used in the school, it was adjusted to the suggestions of the schoolchildren. Also, observation of the children provided feedback on the best methods to use, what needed to be clarified, and what the next level of educational advancement should be. Halff, in "Curriculum and Instruction[1] in Automated Tutors," said that it was necessary to experiment with alternative methods of tutoring, and that:

> Design knowledge can also come from observation. Of interest in this regard are *Wizard-of-Oz* systems, semiautomated tutors in which a human tutor (like the Wizard of Oz) replaces some or all of the instructional functions of an automated tutor (like the machine that the Wizard used to project a wizardly presence to visitors). (Halff, 1988, pp. 99-100)

This study was completed in the same manner: the teacher filled in many gaps between computerized sections. In this way, the students received instruction in many different ways, optimizing the theme of multiple representation beyond the computer screen. Other additions included: verbal instruction, paper construction activities, manipulatives, and a book. The constant presence of a teacher, was for the purpose of adapting the project to each individual student, providing human contact (which should *never* be completely left out of computer instruction), and troubleshooting for program bugs in this still-developing prototype.

In order to provide more social contacts, two students at a time were scheduled. For this reason, two computers (a Macintosh IIci and a Centris) were brought in each week. In this way, at least they would have the company of one other student. They weren't permitted to use the same machine, however, so that they would *each* have the advantage of trying to solve all screens. (Also, it wouldn't have been possible to separate out their individual scores, from the automatic records.) A future possibility would be to let them work as a team, then *only* count the paper-and-pencil test results. After teaching computer classes, the author has decided that a two-person team would be optimal; but it is not always possible, especially when apaptive, individualized instruction is used.

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[1]He defines *curriculum* to mean "the selection of and sequencing of matierial to be presented to students" while *instruction* means "the actual presentation of that material to students" (Halff, p. 80).

Also, the author's presence at all the computer sessions enabled her to encourage and clarify suggestions made by the students, as to how to improve the software. Their expressions of interest and unscheduled time spent on certain portions of the modules was another influence on the modification of the module. Also, several children drew helpful pictures, to show how to create new questions or to modify the present presentation. Several of these were included in the next week's computer session (in which the author pointed how the child's idea had influenced the development of that section).

While observing them, the author also wrote down a few comments the children had made (verbal protocol). A future possibility would be to record their conversations, for analysis of best methods of instruction (both by computer and human interaction), clarification of confusing choices, and indications of where sections should be expanded or shrunk. This was done already, to a limited degree. Future plans have been changed or expanded, due to direct observation of the students: certain repeated displays will be changed into single, random-generated questions (due to unpopularity of this small set); colorful *3d* illustrations will be included even more (due to interest); level of the questions must be adapted closer to the children's abilities, either by expanding *all* the possibilities to include many more gradations or by adjusting them to this narrow range of children; and scoring will be emphasized.

Teaching and Learning from Others

One of the perservering issues that constantly reappeared, throughout development, was the combination of distinct disciplines. Surprisingly, when searching out solutions to programming problems, the author unearthed disparity and lack of communication between several "apparently" similar disciplines. One way this happened was when lack of communication (due to time constraints, narrow focus, and mismatched vocabularies) prevented a commonly used procedure from becoming a solution to a different field's search. For instance, highly sophisticated 3-d modeling systems that slice up data sets into beautifully colored *solid models* and cross-sections are basically unheard of in the field of mathematics education; text-based, incredibly intricate and logical instructional computer programs lack the slickness of accurately animated, physically accurate *front-end* designs--both only located a few buildings, but many perspectives, away: Medical, artistic, and especially architectural software would be of great benefit for mathematics education; more consideration of interface would be make text-based tutors more successful in entering the educational field (generating acceptance by teachers, interest in students, and purchase by administrators); and beautifully graphical forms pursued by commercial artists should be yoked to important content (fulfilling the empty images eagerly broadcast and distributed across the globe).

Meanwhile, on a completely different, more personal level, the teacher's role should be carefully considered in the instructional design. For one thing, the content of the courseware *must* be based on accurate material, gleaned from expert sources. In teaching mathematics, the classroom teacher, the researcher, and the mathematician should be consulted. Since this project was designed to be integrated with regular classroom instruction, it was important to consider the teacher's classroom knowledge and value her suggestions. In this case, the author was lucky enough to have chosen a teacher who was generous both with her classroom and her ideas. For instance, the teacher helped to divide the students into two mathematically even groups. She also made some statistically beneficial suggestions about the design of the research. As to the other considerations (research and mathematics): the author (1) sifted through much related research literature (as seen in the preceding literature review) and (2) studied math and math education (receiving a math teaching certificate).

Integration of many fields is necessary for any instructional courseware design to be successful. Respect for other fields, visiting other laboratories for interviews with experts in related fields (humbly, even going outside one's area of expertise), and openness to innovation should be included by any designer. Many levels of proficiency are needed, for instance in: (1) domain expertise, (2) programming abilities (this *is* a computer program), (3) interface design (communication with the student), (4) strategy and sequencing (storyplot), and (5) evaluation of the student (knowing where to go next). When one is lacking in an area, other people should be sought out.

Returning to the subject of "teaching:" while working with a classroom teacher, there are several reasons to maintain consistently good communication with this teacher. One is that success or failure of the software may be closely linked to the way the teacher is using it. Also, the teacher's suggestions may be as valuable as the student's verbal protocol, in shaping the development of the software. Another reason is that the teacher can give specific hints as to how to deal with this class (the teacher knows them better). Especially if a developer has *not* already done some teaching, the teacher's advice about teaching strategies may be invaluable. Otherwise, learning by experience, which takes longer (perhaps longer than the study), will have to occur. Another way the teacher can be helpful is that he/she can report the students' out-of-classroom, perhaps more candid, comments. The last point, here, is dissemination of the program. If the software is to be distributed through schools, it won't succeed unless the teachers have a positive attitude towards it and understand it. For this purpose, free demonstrations (which includes this type of study) and teacher training should be developed.

Many of these ideas have been supported through research. However, that is not the purpose of this section of the paper. Explaining the rationale, points of emphasis, and design of the pilot are all that could be accomplished at this time. Perhaps another paper should be written to expand upon the particulars of the "teaching" part of the design's development. Meanwhile, the author looks foward to implementing the next stage of the designwork, with the help of a programmer.

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